Excitation-Dependent Frequency Response and Spectral Characterization of Nonlinear SISO Systems

TargetDeLab Technical Paper 05

TargetDeLab Group

2026

Abstract

This study reconstructs the frequency-domain analysis of nonlinear SISO systems so as to avoid the invalid extension of the linear transfer function H(s)=Y(s)/U(s) to a general nonlinear system. Instead of a unique excitation-independent H(j ω), measurable quantities are defined that are explicitly excitation dependent: the fundamental response G1(A,f), the phase of the fundamental component, harmonic components, total harmonic distortion (THD), intermodulation products under two-tone excitation, and - only when the fundamental response is monotonically low-pass - a conditional -3 dB bandwidth $B_{-3mathrmdB}$(A). The methodology is applied to a stable first-order nonlinear system with a cubic term and to a damped Duffing oscillator. The simulations show clear amplitude dependence: for the first-order system $B_{-3mathrmdB}$ shifts from 5.081 Hz at A=0.05 to 9.439 Hz at A=1, while for the Duffing system the resonance peak shifts from 10.462 Hz to 13.846 Hz as amplitude increases from 0.02 to 0.20. With unit excitation at 3 Hz, the first-order system exhibits THD of 7.395%, dominated by the third harmonic, and the Duffing oscillator under two-tone 8/11 Hz excitation produces strong intermodulation products at 5 and 14 Hz. Finally, for A=10^{-3} the nonlinear simulation agrees with the linearization with a maximum relative error below 1.9x10^{-6}. The results demonstrate that frequency-domain characterization of a nonlinear system is feasible, but it must be stated as local or excitation dependent and accompanied by measures of nonlinear spectral generation.

Keywords: nonlinear systems, nonlinear frequency response, fundamental response, harmonics, THD, intermodulation, Duffing, small-signal linearization, conditional bandwidth.

1 Introduction

The classical transfer function is a property of linear time-invariant systems: superposition allows the input-output relation to be represented by convolution and, after a Fourier or Laplace transform, by a unique multiplicative factor H(j ω). In a general

nonlinear system, superposition does not hold. A sinusoidal excitation may generate harmonics, and two tones may produce combination frequencies. Consequently, the pointwise quotient Y(j ω)/U(j ω) from one specific test should not be interpreted, without additional assumptions, as an intrinsic excitation-independent transfer function.

Modern literature has developed several complementary approaches: generalized frequency response functions (GFRFs) and Volterra kernels, OFRF/NOFRF, describing functions, harmonic balance, and best linear approximation (BLA). Cheng et al. [1] summarize the relation between Volterra series and GFRF, NOFRF, and OFRF, while Rijlaarsdam et al. [2] systematically compare the principal families of frequency-domain methods for nonlinear systems. The BLA approach of Schoukens and co-workers [3-5] further shows that even when a linear approximation is used, it depends on the excitation class and statistics and must be accompanied by quantification of nonlinear distortion.

This study adopts a methodology that is rigorous and directly applicable to the numerical objectives: rather than attempting to identify high-dimensional Volterra kernels, it characterizes the periodic steady-state response as a function of excitation amplitude and frequency. This allows Bode-like plots to be retained as useful tools without assigning them the meaning of a universal linear transfer function.

1.1 Periodic Steady State in Nonlinear Systems

For a stable nonlinear system driven by a periodic input, the asymptotic response can contain the excitation frequency together with harmonics and, for multitone inputs, intermodulation products. These additional lines are not numerical errors; they are a direct spectral signature of nonlinearity. If u(t)=A sin(ω t), a polynomial nonlinearity can generate components at integer multiples 2ω, 3ω, and higher. With two excitation frequencies ω_1 and ω_2, combination tones such as $2\omega_1$-ω_2 and $2\omega_2$-ω_1 can appear [1-2,8].

A frequency-domain experiment must therefore specify both the excitation and the output quantity to be retained. The fundamental-response measure used here extracts only the output component at the driving frequency. Harmonic distortion and intermodulation are reported separately so that the discarded nonlinear content remains visible rather than being hidden inside a single Bode-like curve.

1.2 Volterra, GFRF, NOFRF/OFRF, and Describing-Function Perspectives

Volterra series provide a systematic weakly nonlinear extension of linear convolution. The first-order Volterra kernel corresponds to the linear impulse response, while higher-order kernels describe interactions among multiple input frequencies. Fourier transformation of these kernels produces multidimensional generalized frequency response functions. Output-frequency-response formulations reorganize the same information so that nonlinear output spectra can be analyzed more compactly for specified classes of inputs [1,6-7].

Describing functions take a more aggressively reduced approach by retaining a selected harmonic, typically the fundamental, of a nonlinear element under sinusoidal excitation. They can be very useful for approximate resonance and limit-cycle reasoning but do not reconstruct all harmonics. The amplitude-dependent fundamental response G1(A,f) used in this chapter has a similar measurement objective but is defined directly from periodic steady-state simulation rather than assumed to be a universal describing function of an isolated nonlinearity.

1.3 Amplitude-Dependent Resonance and the Duffing Oscillator

The Duffing oscillator is a canonical model of nonlinear stiffness. In its common forced form, a cubic restoring term modifies the resonance curve so that the apparent resonance frequency depends on response amplitude; hardening stiffness bends the resonance toward higher frequencies, whereas softening stiffness bends it toward lower frequencies. At sufficiently strong forcing, multiple periodic responses, jump phenomena, and hysteresis can occur. These effects make the Duffing equation a useful benchmark for demonstrating why a single amplitude-independent frequency-response curve is inadequate [9-10].

The simulations in this chapter use the Duffing model not to reproduce every possible nonlinear-vibration phenomenon but to illustrate one central principle: the measured fundamental response can shift systematically as the excitation amplitude changes. An amplitude-indexed family of curves therefore conveys more information than a single quotient Y/U.

1.4 Harmonic Distortion, Intermodulation, and Conditional Bandwidth

Total harmonic distortion is a normalized measure of the energy in harmonics relative to the fundamental and is meaningful when a fundamental tone is well defined. Intermodulation metrics quantify energy generated at combination frequencies under multitone excitation. Both measures are complementary to a BLA or fundamental response because they quantify spectral content that no single linear frequency response can reproduce [3-5].

The familiar -3 dB bandwidth is also a linear-system concept that presumes a suitable low-pass magnitude shape. In a nonlinear experiment the fundamental response may be nonmonotonic, resonant, or amplitude dependent. The bandwidth reported here is consequently conditional: it is defined only for a particular excitation amplitude and only when the measured fundamental-response curve admits an unambiguous -3 dB crossing.

2 Methodological Validation and Theoretical Reconstruction

2.1 Central Modeling Issue

The study does not reject the idea of frequency-domain characterization. Instead, it replaces the invalid universal concept with measurable quantities that explicitly state their dependence on excitation. The validated MATLAB package does not perform pointwise division of two FFTs to form a "transfer function". It simulates periodic steady state, estimates Fourier coefficients by least-squares harmonic fitting, and reports fundamental gain/phase, THD, and intermodulation products.

2.2 Definition of the Fundamental Excitation-Dependent Response

$$\mathrm{u(t) = A\ sin(\omega t),\ \omega = 2\pi f} \tag{1}$$

For a stable system that reaches a periodic steady state with a period equal to an integer multiple or submultiple of the excitation period, the output can be written as a Fourier series:

$$y(t) = y_0 + \sum_{h=1}^{\infty} Y_h(A,\omega)\sin(h\omega t + \phi_h(A,\omega)) \tag{2}$$

Define the complex fundamental response for this family of sinusoidal excitations as:

$$G_1(\mathrm{A},\omega) = [\mathrm{Y_1(A},\omega)/\mathrm{A}]\ \exp(\mathrm{j}\varphi_1(\mathrm{A},\omega)) \tag{3}$$

Unlike an LTI FRF, G_1 depends on amplitude A. This dependence is not a measurement defect but a fundamental feature of the nonlinear system. The small-signal claim follows from first-order perturbation theory. Let the nonlinear system be $\dot{x} = f(x,u)$, $y = h(x,u)$, with equilibrium (x_0,u_0) and continuously differentiable f,h. For $u = u_0 + A\tilde{u}$ and $x = x_0 + A\xi + O(A^2)$, Taylor expansion gives $\dot{\xi} = F_x\xi + F_u\tilde{u} + O(A)$ and $\tilde{y} = H_x\xi + H_u\tilde{u} + O(A)$ after division by A. Hence, as $A \to 0$, the normalized first-harmonic input-output map tends to the LTI linearization $G_{\mathrm{lin}}(s) = H_x(sI - F_x)^{-1}F_u + H_u$, provided the linearized equilibrium is well posed and the periodic steady state exists. Higher-order harmonics scale with higher powers of A and therefore vanish relative to the fundamental. Thus $\lim_{A\to 0} G_1(A,\omega) = G_{\mathrm{lin}}(j\omega)$ under these smoothness and stability assumptions.

2.3 Harmonic Distortion and Intermodulation

$$\mathrm{THD}(A,\omega) = \frac{\sqrt{\sum_{h=2}^{H} Y_h^2(A,\omega)}}{Y_1(A,\omega)} \tag{4}$$

THD measures the portion of the periodic response that does not belong to the fundamental component. For systems with odd symmetry and zero bias, such as a cubic nonlinearity, predominantly odd harmonics are expected. With the two-tone input $u(t) = A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$, a cubic term generates, among others, the third-order combination frequencies 2 ω_1-ω_2 and 2 ω_2-ω_1. These new spectral lines are direct evidence that no scalar H(ω) can merely map frequencies already present in the input to the output.

2.4 Conditional Bandwidth

A bandwidth can be defined only relative to a particular response characteristic and a particular excitation. For a monotonically low-pass fundamental response, define:

$$B_{-3\mathrm{dB}}(\mathrm{A}) = \inf \{ \mathrm{f} : |G_1(\mathrm{A},\mathrm{f})| \leq |G_1(\mathrm{A}, f_{\mathrm{ref}})| / \sqrt{2} \} \quad (5)$$

where f_{ref} is a sufficiently low reference frequency. $B_{-3mathrmdB}$(A) is therefore an amplitude-conditioned bandwidth. It is not used as a universal property of systems exhibiting resonance, multiple peaks, jump phenomena, or other nonmonotonic behavior.

3 Simulation Models

Excitation/measurement topology. This chapter is primarily an open-loop characterization: prescribed single-tone or two-tone u(t) excites either nonlinear plant, and steady-state output data are analyzed to obtain the fundamental response, harmonic distortion, intermodulation products, and conditional bandwidth.

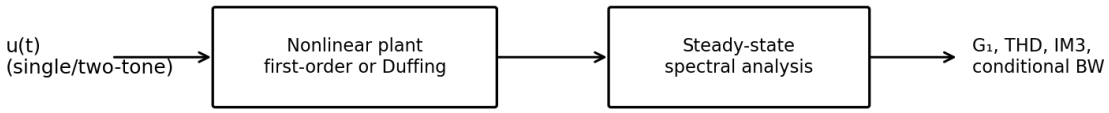


Figure 33. Control topology used in Chapter 5.

3.1 Stable First-Order Nonlinear System

$$\dot{x} + \mathrm{ax} + \alpha \mathrm{x}^3 = \mathrm{bu},\ \mathrm{a}=\mathrm{b}=2\pi\cdot 5,\ \alpha=80 \quad (6)$$

The linear part has unit DC gain and a cutoff frequency of 5 Hz. The cubic term is dissipative with respect to the magnitude of x and ensures a well-defined bounded response for the periodic excitations considered. The local linearization about x=0 is:

$$H_{\text{lin}},1(s) = b/(s+a) \tag{7}$$

3.2 Damped Duffing Oscillator

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0{}^2x + \alpha x^3 = \beta u \tag{8}$$

$$f_0=10 \text{ Hz}, \omega_0=2\pi f_0, \zeta=0.08, \alpha=2\times10^4, \beta=\omega_0{}^2 \tag{9}$$

The positive cubic stiffness produces hardening behavior, so the resonance peak is expected to shift toward higher frequencies as amplitude increases. The linearization is:

$$H_{\text{lin}},2(s) = \beta/(s^2 + 2\zeta\omega_0 s + \omega_0{}^2) \tag{10}$$

4 Numerical Methodology

The simulations are performed with MATLAB ode45 using tight tolerances RelTol=2x10^{-9} and AbsTol=10^{-11}. For each tone, 35 periods are simulated and the first 25 are discarded as transient response. Over the final periods, the output is fitted with a linear least-squares model containing a DC term and pairs cos(h ω t), sin(h ω t). Fourier coefficients are therefore estimated without the spectral leakage that an arbitrary FFT window would introduce. For the two-tone test, a Hann window is used after transient removal and spectral lines up to 40 Hz are analyzed.

Independent reproduction of the study numerical results used equivalent high-accuracy integration and harmonic least-squares fitting.

5 Simulation Scenarios

Table 23. 5.5. Simulation Scenarios.

ID	Test	Objective
1	First-order single-tone sweep, A={0.05,0.2,0.5,1}	Amplitude dependence of G_1 and $B_{-3\text{dB}}$
2	Single-tone sweep Duffing, A={0.02,0.08,0.20}	Resonance shift / hardening
3	First-order: A=1, f=3 Hz	Harmonics and THD

4	Duffing: 8 Hz + 11 Hz, A=0.12 per tone	Intermodulation
5	First-order: A=10^{-3}	Convergence to the linearization

6 Results

6.1 Scenario 1 - Amplitude Dependence of the Fundamental Response

Scenario definition. The stable first-order nonlinear plant is excited by single sinusoids over the frequency sweep using amplitudes A=0.05, 0.2, 0.5, and 1.0. For each tone 35 periods are simulated, the first 25 are discarded, and the retained steady state is fitted harmonically to obtain G1(A,f), THD, and the conditional -3 dB bandwidth.

The G_1(A,f) curve of the first-order system is low-pass for all examined amplitudes, but it is not the same curve. The cubic action changes both the low-frequency gain and the roll-off frequency. The conditional $B_{-3mathrmdB}$ increases systematically with amplitude.

Table 24. Results and parameters.

Amplitude A	**B-3dB(A) [Hz]**
0.05	5.081
0.2	5.516
0.5	6.985
1.0	9.439

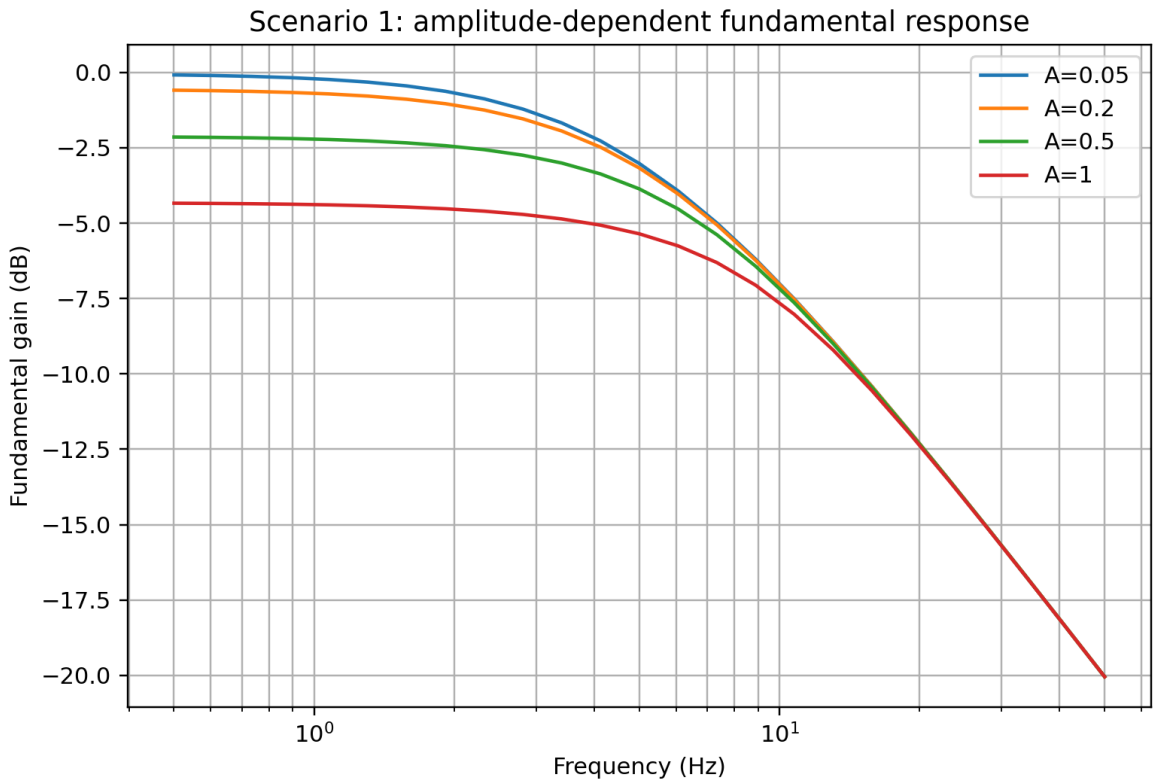


Figure 37. Fundamental response of the first-order system for four excitation amplitudes.

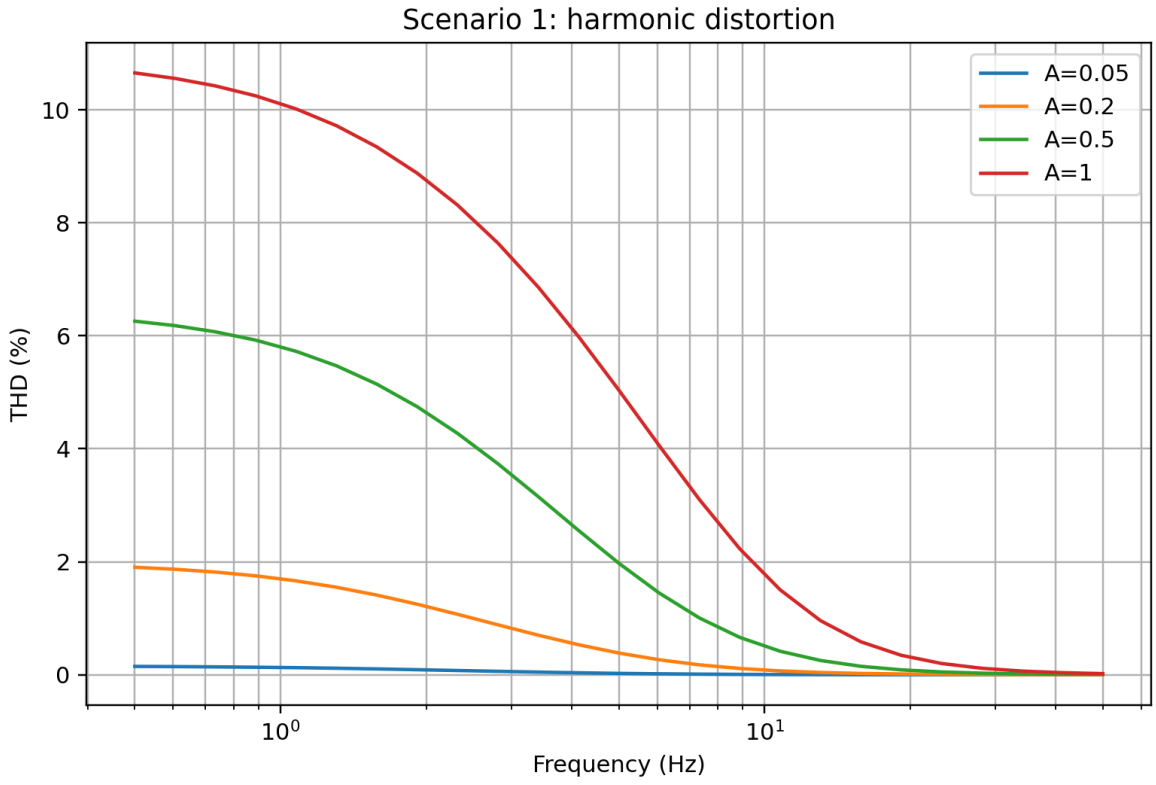


Figure 38. THD as a function of frequency and amplitude for the first-order system.

The bandwidth change from 5.081 Hz to 9.439 Hz should not be interpreted as four different “transfer functions” of the same LTI system. It is precisely the information that is lost if nonlinear behavior is collapsed into one H(j ω): the system has a family of responses parameterized by amplitude.

6.2 Scenario 2 - Amplitude-Dependent Duffing Resonance

Scenario definition. The damped hardening Duffing oscillator is swept with single-tone inputs at amplitudes A=0.02, 0.08, and 0.20. The same steady-state extraction procedure is used, and the resonance peak frequency and peak fundamental gain are recorded to quantify amplitude-dependent hardening.

Table 25. 5.6.2. Scenario 2 - Amplitude-Dependent Duffing Resonance.

A	**fpeak [Hz]**	G_1**,peak**	G_1**,peak [dB]**
0.02	10.462	5.782	15.241
0.08	12.154	4.747	13.528
0.2	13.846	2.770	8.851

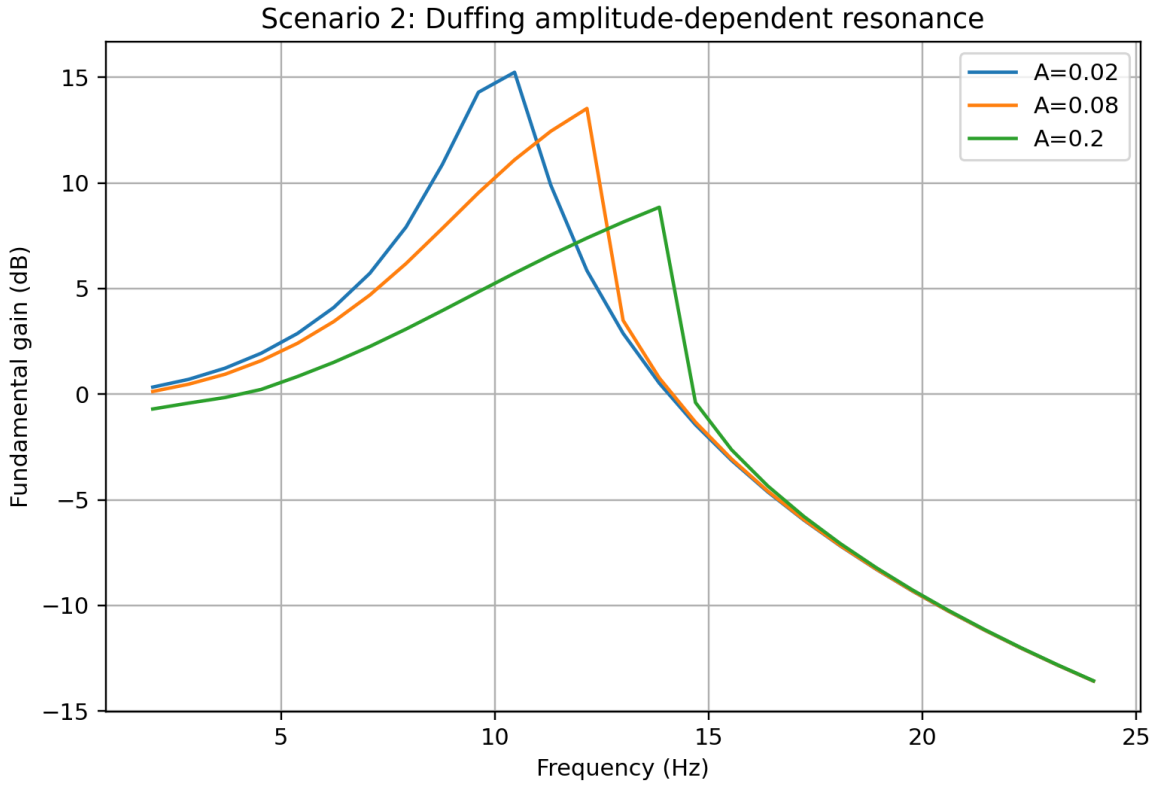


Figure 39. Duffing fundamental response. The peak shifts toward higher frequency as amplitude increases.

The peak shifts from 10.462 Hz to 12.154 Hz and 13.846 Hz for A=0.02, 0.08, and 0.20, respectively. At the same time, peak gain decreases from 15.24 dB to 8.85 dB. This is a typical indication of hardening nonlinearity and provides a second, independent demonstration that a single amplitude-independent Bode curve is insufficient.

6.3 Scenario 3 - Harmonic Generation

Scenario definition. The first-order nonlinear plant is driven by the pure sinusoid u(t)=sin(2 pi 3t), i.e. A=1 at 3 Hz. The output fundamental and odd harmonics are measured after transient removal; THD is computed from the harmonic content.

For an input of A=1 at 3 Hz, the fundamental output amplitude is 0.5784 and THD=7.395%. The third harmonic is 7.301% of the fundamental, the fifth is 1.156%, the seventh is 0.206%, and the ninth is 0.0383%. Even harmonics remain at the numerical noise floor, as expected from the odd cubic symmetry of the model.

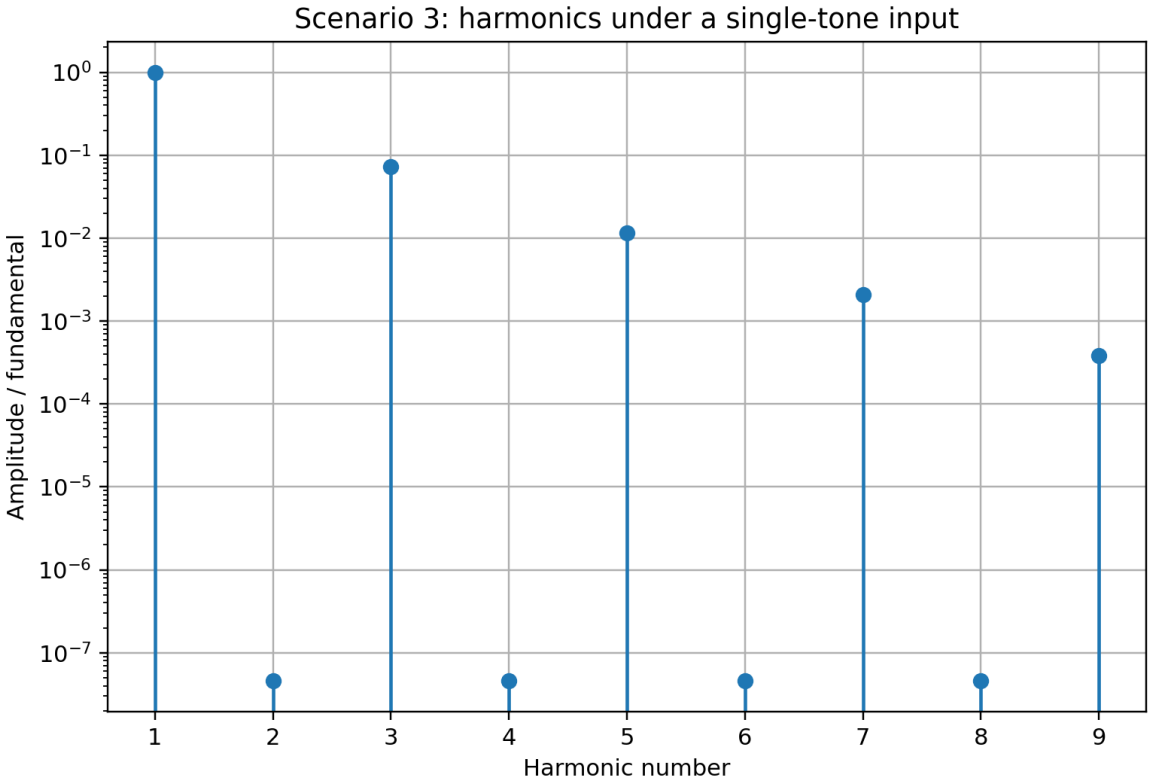


Figure 40. Normalized output harmonics for pure 3 Hz sinusoidal excitation.

6.4 Scenario 4 - Two-Tone Intermodulation

Scenario definition. The Duffing oscillator is driven simultaneously at 8 and 11 Hz with amplitude 0.12 per tone. After transient removal a Hann-window spectrum is evaluated up to 40 Hz. The unexcited third-order products at 5 Hz and 14 Hz are used as direct intermodulation indicators.

With 8 and 11 Hz inputs and amplitude 0.12 per tone, the unexcited lines $2f_1$-f_2=5 Hz and $2f_2$-f_1=14 Hz appear in the output. Their amplitudes are 0.02256 and 0.11024, respectively, compared with 0.08836 at 8 Hz and 0.29450 at 11 Hz. The IM3 ratios are 25.53% and 37.43%. These spectral lines cannot be generated by any linear scalar H(j ω) when the input contains neither 5 nor 14 Hz.

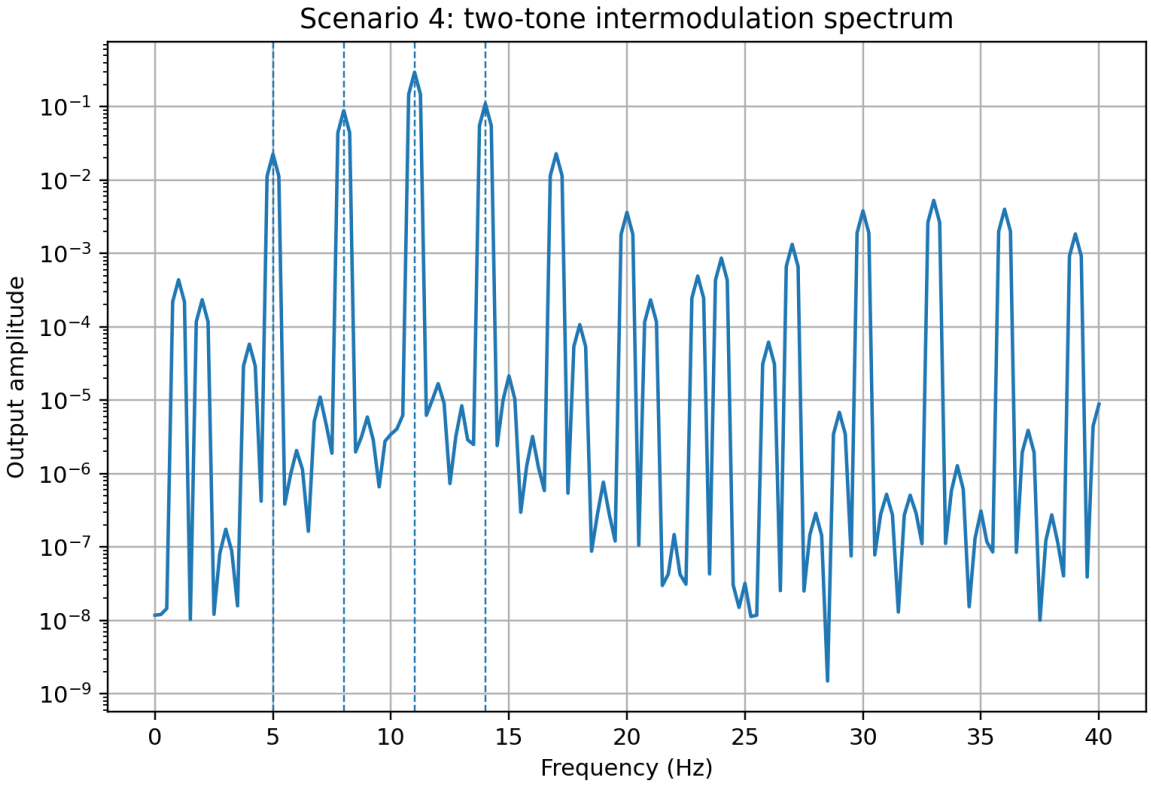


Figure 41. Duffing output spectrum under two-tone 8/11 Hz excitation, showing intermodulation products.

6.5 Scenario 5 - Small-Signal Limit

Scenario definition. The first-order nonlinear plant is excited with very small single-tone amplitude A=10^{-3} over 0.5-50 Hz. Its measured fundamental response is compared directly with the local linearization H_{lin},1(j ω) to verify convergence to the classical small-signal FRF.

For A=10^{-3}, the nonlinear fundamental response of the first-order system is compared with H_{lin},1(j ω)=a/(j ω+a). The maximum relative error over 0.5-50 Hz is 1.872e-06 and the RMS relative error is 1.072e-06. The numerical agreement verifies that the new definition G_1(A,f) connects smoothly to the classical linear FRF in the small-signal limit.

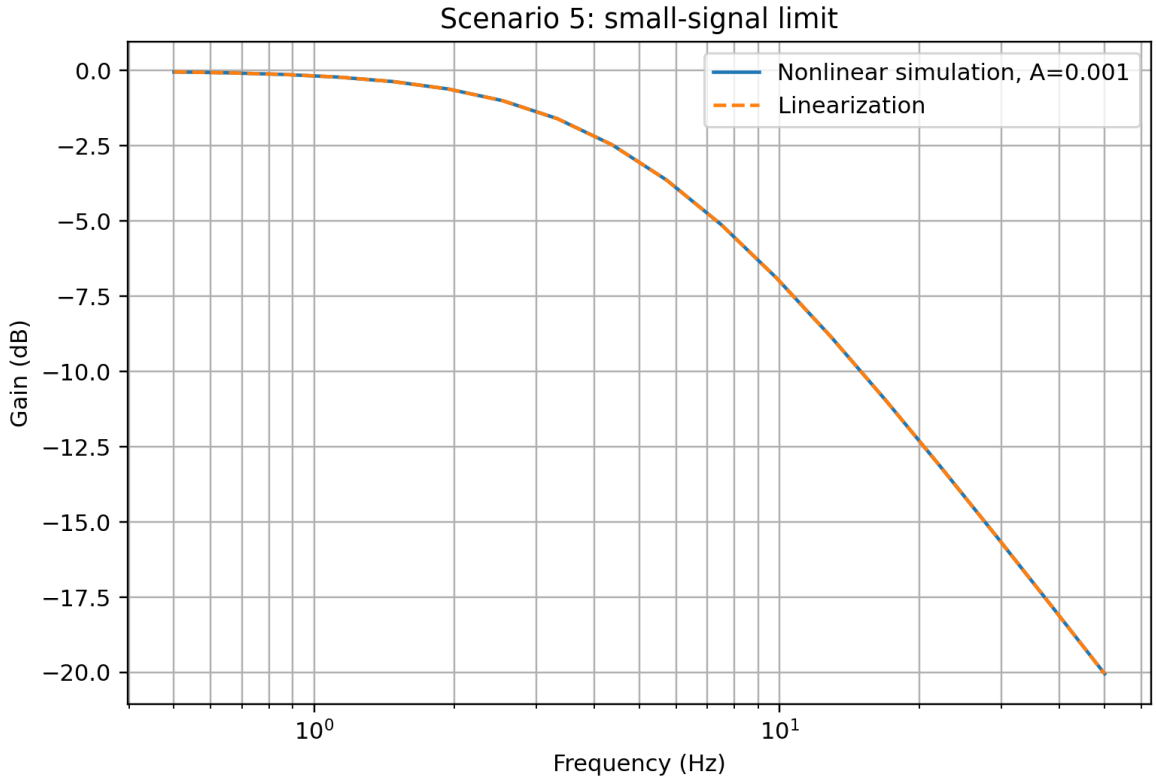


Figure 42. Comparison of the small-amplitude nonlinear simulation with the local linearization.

7 Discussion

The reconstruction substantially changes the interpretation, not the central engineering idea that frequency-domain analysis can be useful for nonlinear systems. The key point is that frequency-domain information must be multidimensional or explicitly conditioned on the experimental condition. In its simplest form, the family G_1(A,f) provides a Bode-like tool that is easy to measure and use. When a more global description of nonlinear interactions is required, the natural extension is to use GFRFs/Volterra kernels or OFRF/NOFRF [1,6].

Scenario 1 shows why bandwidth should not be reported without an amplitude tag. The Duffing result shows even more strongly that frequency-domain shape can change qualitatively with amplitude. Scenarios 3 and 4 show the information that G_1 cannot contain: harmonics and intermodulation. For this reason, the proposed procedure never reports gain/phase alone, but accompanies G_1 with THD or targeted multitone analysis.

The BLA is an alternative when the objective is to extract the best linear approximation for a specified class of random/multitone excitation. The relevant theory emphasizes that the estimated linear behavior and the nonlinear distortions must be examined together [3-5]. This study does not claim that G_1 is a BLA: it is a deterministic single-tone fundamental response. This distinction is deliberate and should be

retained in future experimental applications.

8 Conclusions

The methodology is formulated so that the frequency-domain analysis is consistent with nonlinear-systems theory. The quotient of output FFT to input FFT is no longer used as a universal “transfer function”. Instead, an excitation-dependent fundamental response G_1(A,f) is defined together with phase, harmonics, THD, and intermodulation metrics. Where monotonically low-pass behavior exists, an amplitude-conditioned bandwidth $B_{-3mathrmdB}$(A) is defined.

The numerical tests provide five independent validations of the new formulation: (i) the bandwidth of the first model changes with amplitude, (ii) the Duffing oscillator shifts resonance toward higher frequencies, (iii) single-tone excitation generates harmonics, (iv) two-tone excitation generates new intermodulation frequencies, and (v) in the small-signal limit the fundamental response converges to the classical linear FRF. Frequency-domain characterization of nonlinear SISO systems is therefore highly useful provided the type of approximation and its dependence on input and operating point are stated explicitly.

Taken together, these tests separate three effects that a single Bode-like curve would mix together: amplitude-dependent change of the fundamental response, generation of harmonics under single-tone excitation, and generation of new intermodulation frequencies under multitone excitation. The small-signal test then closes the loop with classical theory by showing that these nonlinear descriptors reduce to the local linear FRF as the excitation amplitude tends to zero. The practical conclusion is not that classical frequency response should be abandoned, but that its validity must be demonstrated for the amplitude and excitation class of interest.

Relationship to the Complete Book

This paper is a self-contained study extracted and reformatted from the corresponding research chapter of *Nonlinear Control: From Model to Evidence*. The complete book provides the unified progression across all seven studies, cross-study synthesis, extended methodological context, and the MATLAB/Octave and Python software-companion architecture used for replication and further experimentation.

8.1 References

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