Exact Nonlinearity Cancellation and an Equivalent Linear System in Closed-Loop Control

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Abstract

A specific but important class of higher-order nonlinear SISO systems is studied in which the input enters additively and the nonlinearity is matched to the input channel. For a system $x^{(L)} = f(x) + u$, define u=v-$\hat{f}(x)$. When $\hat{f} = f$, the physical system satisfies exactly $x^{(L)}$=v and therefore, for zero initial conditions, the transfer from v to x is $1/s$^L. The Linear Equivalent System (LES) therefore refers to the actual plant output x rather than to an auxiliary variable. Generalized PID controllers, Hurwitz conditions, a derivative-free integral least-squares method for estimating polynomial f(x), correct error/energy metrics, and numerical verification are presented. For the system ẍ=-x^2+x^3+u, exact cancellation coincides numerically with the LES to machine precision, while coefficient estimation with relative error 7.33x10^{-4} gives RMSE 1.14x10^{-4} in the tracking scenario.

Keywords: nonlinear control, feedback linearization, exact cancellation, dynamic inversion, PID, integral least squares, nonlinear system identification, matched uncertainty.

1 Introduction

Nonlinear dynamical systems generally do not possess a unique transfer function independent of operating point and input. Classical linear techniques such as Bode plots, pole placement, and loop shaping therefore cannot be applied directly to the nonlinear model. For specific system structures, however, exact transformation of the dynamics by feedback is possible. This area is known as feedback linearization/exact linearization and is an established part of nonlinear control theory [1-4].

This study is explicitly restricted to the class of scalar SISO systems in which the input enters additively with known sign and the nonlinearity is matched to the same channel. This qualification is critical: the method does not claim universal linearization of arbitrary nonlinear systems. For the structure considered here, however, cancellation is exact when the nonlinear model is exact.

1.1 Feedback Linearization and Exact Nonlinearity Cancellation

Feedback linearization transforms a nonlinear control problem into a linear one by algebraically cancelling known nonlinear terms and introducing a new virtual input. For a general affine system $\dot{x}$=f(x)+g(x)u, exact state or input-output linearization requires structural conditions involving relative degree, the decoupling matrix, and, when the relative degree is smaller than the state dimension, the stability of internal or zero dynamics [1-4,11]. The scalar system treated here is deliberately simpler: the input enters the highest derivative with unit coefficient, so the cancellation can be written directly without a nonlinear coordinate transformation.

Exact cancellation must be interpreted cautiously. It is a model-based operation and is therefore only exact if the nonlinear term is known accurately and the commanded input is actually realized by the actuator. Model mismatch, unmodeled dynamics, sensor error, and saturation reintroduce nonlinear or uncertain terms into the transformed dynamics. Robust feedback linearization and adaptive control address these effects by estimating uncertainty or designing the residual closed loop to tolerate it [2-4].

1.2 Relative Degree, Zero Dynamics, and the Scope of the Equivalent Linear System

In a general SISO nonlinear system, differentiating the output until the input appears defines the relative degree. If the relative degree equals the state dimension, the input-output linearizing coordinates can represent the full local state and there are no internal zero dynamics. If the relative degree is smaller, the input-output channel can be linearized while internal dynamics remain. Stability of those zero dynamics is then essential for a stable overall closed loop [1,11].

The 'equivalent linear system' used in this chapter is therefore a special-case construction, not a claim that an arbitrary nonlinear plant can always be converted into a globally equivalent LTI system. Its value lies in making the assumptions visible: the derivative order, the nonvanishing input coefficient, the availability of the required state or output derivatives, and the fidelity of the nonlinear cancellation model.

1.3 Identification from Data and Why Weak/Integral Forms Matter

When f(x) is not known analytically, cancellation requires a model identified from data. Direct differentiation of noisy measurements is notoriously ill conditioned because high-frequency noise is amplified by differentiation. Integral, weak-form, and modulating-function identification methods avoid this problem by transferring derivatives from measured signals to chosen test functions through integration by parts. The result is a regression in which unknown model parameters appear algebraically while the measured signal itself remains undifferentiated [5-8,12].

The integral least-squares method used in this chapter follows this general philoso-

phy. When f(x) is represented in a chosen basis, integration over windows produces a linear regression for the basis coefficients. The method does not eliminate model-selection error: a basis that is too small causes bias, whereas an unnecessarily large basis can increase variance and numerical conditioning problems. Independent validation of the identified model is therefore required before it is used in a cancellation law [9,13-14].

1.4 Inversion, Feedforward, and the Difference Between Exact Tracking and Optimality

Model inversion determines an input that reproduces a desired trajectory under a specified model. Optimal control, by contrast, minimizes a cost functional subject to system dynamics and constraints. An inverse input can coincide with an optimal solution for a specially chosen cost and admissible set, but this is not automatic. The exact-trajectory feedforward term in this chapter is therefore described as the input required by the nominal inverse model rather than as a universal minimum-energy input.

This distinction is particularly important when actuator bounds or tracking tolerances are present. A constrained optimizer may deliberately allow tracking error to reduce peak actuation or quadratic effort. Conversely, exact inversion can demand very large inputs when the reference contains sharp corners or high-frequency content. The triangular-reference scenario later in the chapter is designed to expose precisely this limitation.

2 System Class and Assumptions

Closed-loop topology with the linear converter. The tracking controller produces the virtual linear input v. The converter forms the physical plant input u=v-$\hat{f}(x)$, cancelling the modeled nonlinearity. Under exact cancellation the plant input-output channel from v to x is the equivalent L-fold integrator (LES); model mismatch appears as a residual disturbance.

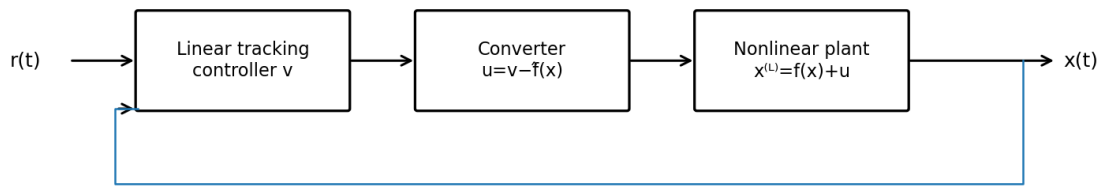


Figure 40. Control topology used in Chapter 6.

Consider a system of order L>=1

$$d^{L}x(t)/dt^{L}= f(x(t)) + u(t) \tag{1}$$

where x in R is the measured output, u in R is the physical input, and f:R->R is continuous and sufficiently differentiable for a unique solution to exist over the operating region. For the polynomial case

$$f(x) = \sum_{m=0}^{k} p_m x^m \tag{2}$$

with real coefficients p_m. The input has a known unit coefficient. For the general system $x^{(L)}$=f(x)+b(x)u, b(x) must be nonzero in the operating region and u=(v-$\hat{f}(x)$)/$\hat{b}$(x); this generalization is not used in the numerical tests.

Equation (1) is a special canonical form with relative degree L with respect to output x. There are no hidden internal dynamics (zero dynamics) in this same scalar representation. This property permits exact cancellation of the matched nonlinearity.

3 Linear Converter and LES

3.1 Exact Input-Side Cancellation

Define a new virtual control signal v(t), and let the Linear Converter generate the physical input

$$u(t) = v(t) - \hat{f}(x(t)) \tag{3}$$

where $\hat{f}$ is the available model of the nonlinearity. If $\hat{f} = f$, substitution of (3) into (1) gives

$$x^{(L)}(t) = f(x) + v - f(x) = v(t) \tag{4}$$

The equivalence now concerns the actual output x(t). For zero initial conditions, the Laplace transform of (4) gives

$$LES(s) = X(s)/V(s) = 1/s^{L} \tag{5}$$

The “Linear Equivalent System” is therefore not an independent substitute for the physical plant, but the exact input-output dynamics of the plant after nonlinear feedback has been applied. This formulation coincides with the classical idea of feedback linearization/nonlinear dynamic inversion for the special form in (1) [1-4].

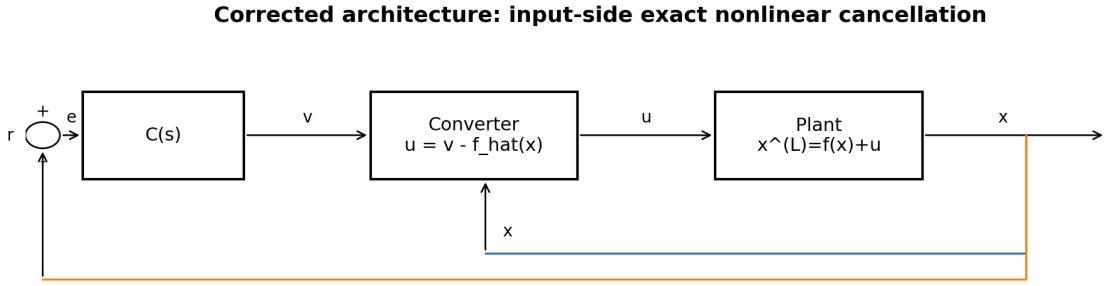


Figure 44. Linear Converter architecture: the converter acts at the input of the physical plant.

3.2 Imperfect Knowledge of the Nonlinearity

If $\hat{f}$ is not equal to f, define the model error Δf(x)=f(x)-$\hat{f}(x)$. Then

$$x^{(L)}(t) = v(t) + \Delta f(x(t)) \tag{6}$$

The nonlinearity is not completely removed but is converted into a matched disturbance. Exact equivalence with $1/s$^L therefore holds only for Δf=0. For bounded Δf, performance depends on the robustness of the outer linear controller and on the region in which the trajectories remain bounded. No global stability claim is made without additional assumptions.

4 Identification of f(x) from Data

4.1 Known Analytical Model

When the coefficients p_m are known, $\hat{f}$ is used directly in the Converter. This corresponds to model-based exact cancellation.

4.2 Integral Least-Squares Method Without Numerical Differentiation

Direct estimation of f from $x^{(L)}$-u requires the Lth numerical derivative of the measurement and can strongly amplify noise. An integral formulation is therefore used, related to modulating-function and weak/integral identification methods that avoid differentiating noisy data [5-8].

Define the L-fold integration operator

$$\mathcal{I}_L[q](t) = \frac{1}{(L-1)!} \int_0^t (t-\tau)^{L-1} q(\tau)\, d\tau \tag{7}$$

and apply the operator to (1):

$$x(t) - \sum_{j=0}^{L-1} \frac{x^{(j)}(0)t^j}{j!} - \mathcal{I}_L[u](t) = \sum_{m=0}^{k} p_m \mathcal{I}_L[x^m](t) \tag{8}$$

For sampled data, (8) gives the linear system $Y = \Phi p$ and the estimate

$$\hat{p} = \text{arg min_p } ||\Phi\text{p-Y}||_2{}^2 \tag{9}$$

which can be computed using QR/SVD or normalized least squares. If the initial derivatives are unknown, the corresponding terms may be introduced as additional linear unknown parameters rather than differentiating x(t).

5 Controller for the LES

A linear controller can be used for the exactly linearized system $1/s$^L. For L>=2, a generalized PID with derivatives up to order L-1 is selected:

$$C_L(s) = \frac{K_I}{s} + K_P + K_{D1}s + K_{D2}s^2 + \cdots + K_{D,L-1}s^{L-1} \tag{10}$$

The closed-loop characteristic equation is

$$s^{L+1} + K_{D,L-1}s^L + \cdots + K_{D1}s^2 + K_P s + K_I = 0 \tag{11}$$

For L=2, C(s)=K_P+K_D s+K_I/s and

$$P_{\text{cl}}(\text{s}):\ \text{s}^3 + K_D\ \text{s}^2 + K_P\ \text{s} + K_I = 0 \tag{12}$$

The necessary and sufficient Hurwitz conditions for a monic cubic polynomial are K_D>0, K_P>0, K_I>0, and K_D K_P>K_I. For L=3, with a second derivative term K_D2, the characteristic polynomial is $s^4 + K_{D2}s^3 + K_{D1}s^2 + K_P s + K_I$ and the full Routh-Hurwitz table applies. The notation “PID(L-1)” is used here to indicate the maximum derivative order, not a fundamentally new controller type.

5.1 Two Degrees of Freedom for Tracking

For a reference r(t) with an available Lth derivative, tracking is improved by feedforward of r^(L):

$$v = r^{(L)} + K_P e + K_{D1}\dot{e} + \cdots + K_{D,L-1}e^{(L-1)} + K_I \int e(t)\,dt \tag{13}$$

For exact cancellation and compatible initial conditions, the error is governed exclusively by the designed linear characteristic polynomial. In the L=2 simulations, poles -8, -10, and -12 are used, giving K_D=30, K_P=296, and K_I=960.

5.2 “Exact-Trajectory Feedforward” Instead of “Minimum Energy”

If ideal tracking $x(t) = r(t)$ is required, (1) uniquely imposes the compatible input

$$u_{\text{ff}}(t) = r^{(L)}(t) - f(r(t)) \tag{14}$$

Equation (14) is not, in general, the solution to a minimum-energy problem and is not a lower bound for every controller. It is the exact feedforward input that realizes the specified trajectory when the model is exact. Only E_{ff}=integral u_{ff}^2 dt can be used as an effort reference for this ideal trajectory.

Moreover, for L=2 an ideal triangular reference is not C^2: its second derivative contains impulsive terms at the corners. Thus integral u_{ff}^2 dt is not finite in the classical sense, and any numerical value depends on the sampling step. Study therefore uses a smooth reference for energy comparisons and retains the triangular reference only as a tracking stress test.

6 Simulation Scenarios

Scenario definitions. All five principal tests use T=4 s and the smooth periodic reference $r(t) = 2[1 - \cos(\pi t)] + 0.75[1 - \cos(3\pi t)]$, which has compatible zero endpoint position/velocity conditions. S1 applies the designed controller directly to the LES $1/s$^2 benchmark; S2 uses the nonlinear plant with exact converter; S3 perturbs the converter coefficients by +/-10%; S4 removes the converter; and S5 constructs the converter from the integral least-squares estimate. The L=2 controller uses poles -8,-10,-12, giving K_D=30, K_P=296, K_I=960. The triangular reference discussed later is a separate stress test rather than the energy-comparison reference.

The numerical example uses the nonlinear term

$$\ddot{x} = -x^2 + x^3 + u \tag{15}$$

and examines five main scenarios over T=4 s. The principal smooth reference is periodic C-infinity, satisfies r(0)=$\dot{r}(0)$=r(T)=$\dot{r}(T)$=0, and is defined by

$$r(t) = 2[1-\cos(\pi t)] + 0.75[1-\cos(3\pi t)] \tag{16}$$

Python simulation was used as an independent numerical verification of the MATLAB implementation with tight integration tolerances. The MATLAB package uses ode45 and the same equations and parameters.

Table 26. Results and parameters.

ID	Scenario	Purpose
S1	LES $1/s^2$	Linear reference benchmark
S2	Nonlinear plant + exact Converter	Exact-equivalence verification

S3	Converter with ±10% coefficient error	Robustness to model mismatch
S4	Without Converter	Baseline effect of nonlinearity
S5	Converter from integral-LS estimate	Data-driven application

7 Numerical Results

Table 27. 6.8. Numerical Results.

Scenario	**IAE**	**ISE**	**RMSE**	**max|e|**	**$\int u^2 dt$**
LES	8.23874e-15	3.36564e-29	2.90071e-15	1.28786e-14	28208.9
Exact converter	8.23874e-15	3.36564e-29	2.90071e-15	1.28786e-14	28208.9
10% model mismatch	0.0540535	0.00119615	0.0172927	0.0391897	27366.7
No converter	0.438149	0.0908689	0.150722	0.351214	36660.2
Identified converter	0.000343488	5.22154e-08	0.000114253	0.000265253	28203.3

In S1 and S2, the difference x_{NL}-x_{LES} is numerically zero in the solver used (maximum absolute difference 0 in stored double precision), as required by the algebraic cancellation. The nearly zero tracking metrics are not a “magical” property of the PID; they result from the combination of exact cancellation, feedforward $\ddot{r}$, and compatible initial conditions.

This equality is the numerical signature of the algebraic cancellation derived earlier. It verifies that, under the exact-model and unsaturated assumptions, the nonlinear physical plant with the Converter and the LES are not merely similar approximations: they produce the same stored trajectory in the implemented solver. The result also explains why the tracking errors in these cases are extremely small; the feedforward term supplies the reference acceleration while the feedback controller corrects residual numerical/tracking error.

With a 10% error in the cancellation coefficients, RMSE increases to 1.729x10^{-2} and max|e| to 3.919x10^{-2}, remaining small. Without the Converter, RMSE becomes 1.507x10^{-1} and max|e| 3.512x10^{-1}. The experiment therefore does not prove general robust stability, but it clearly shows the effect of residual Δf within the specified operating region.

The model-error and no-Converter cases establish a useful performance ordering. Exact cancellation gives the reference behavior, imperfect cancellation leaves a matched residual disturbance and causes a moderate increase in error, and removing the Converter produces the largest error of the three. Because this comparison is made over one operating region and one uncertainty level, it demonstrates sensitivity to model fidelity rather than a general robustness theorem.

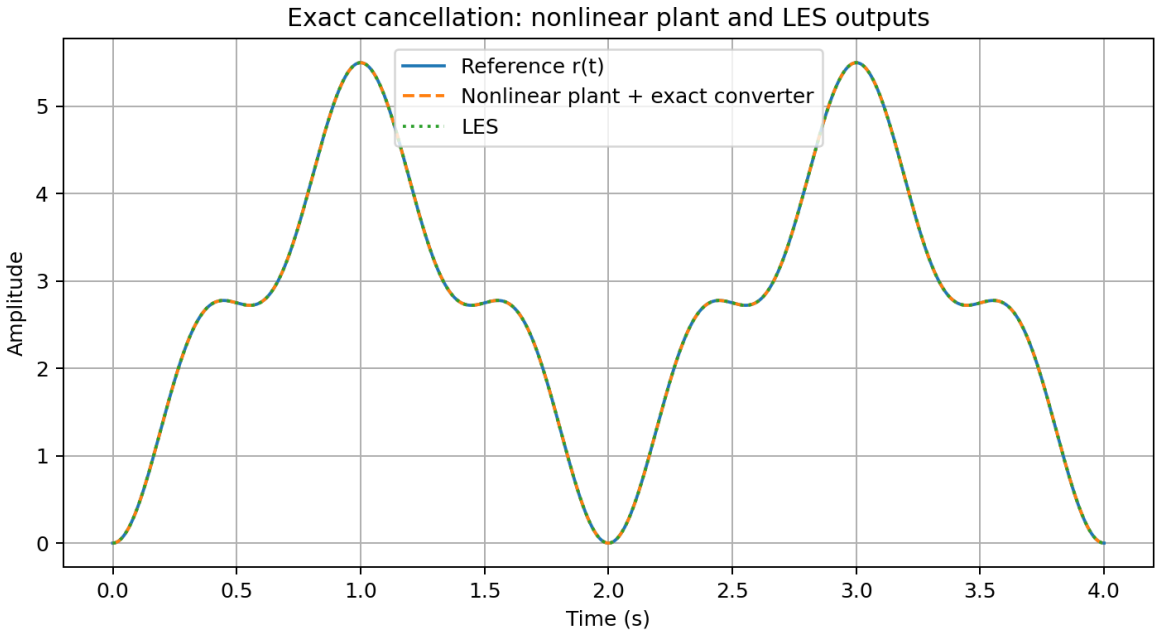


Figure 45. Exact equivalence between the nonlinear plant with Converter and the LES.

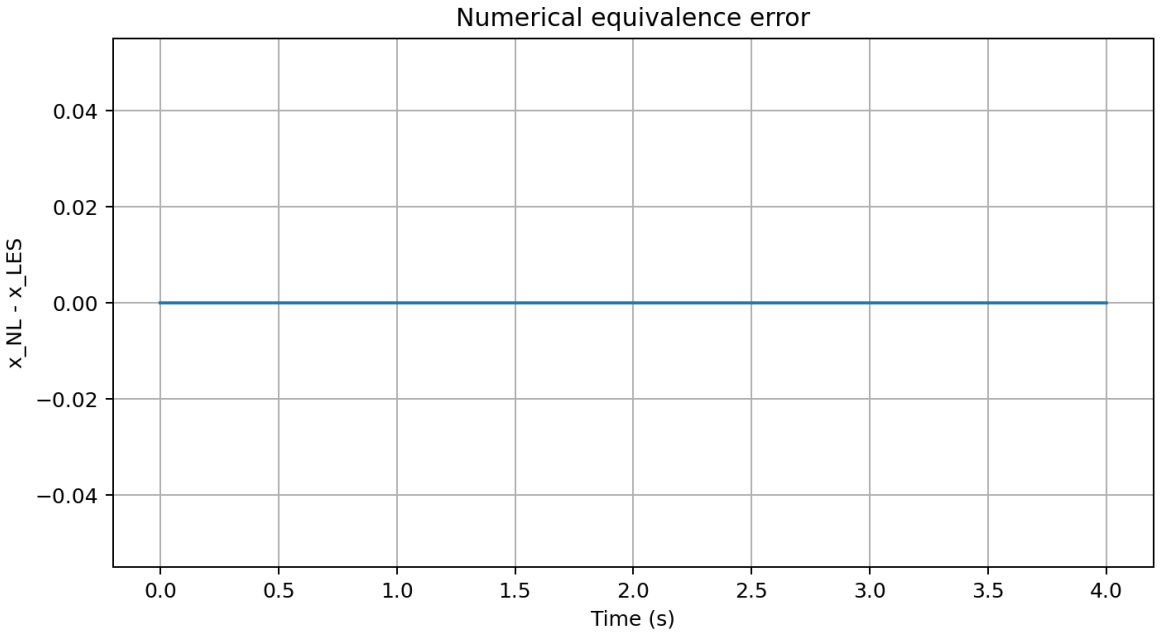


Figure 46. Numerical equivalence error x_{NL}-x_{LES}.

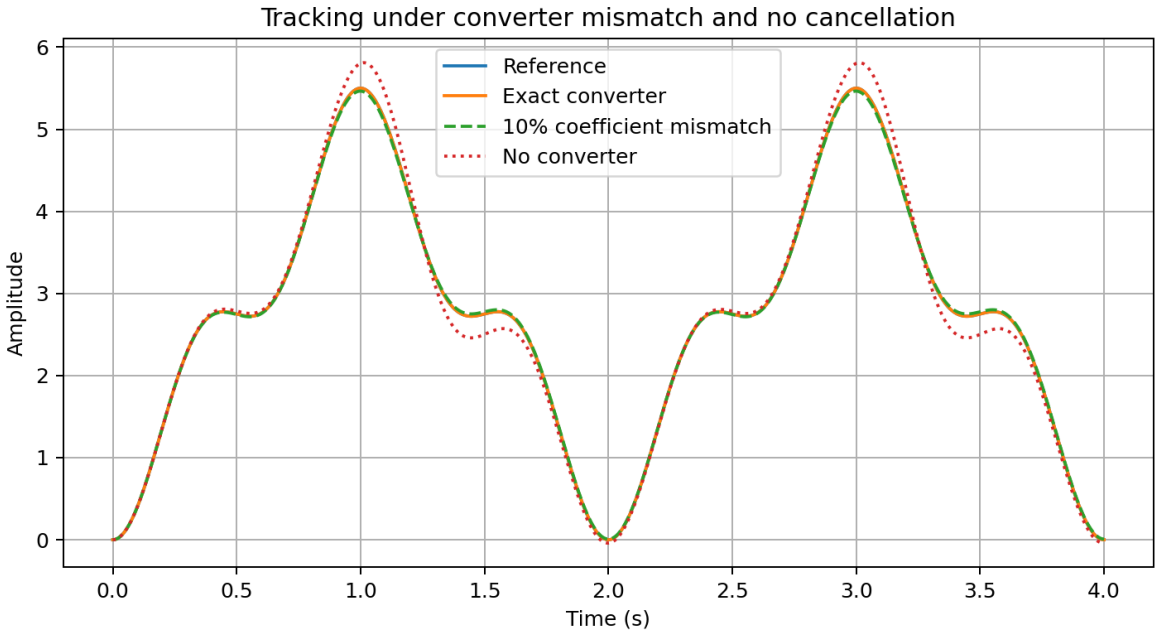


Figure 47. Effect of 10% model error and absence of the Converter.

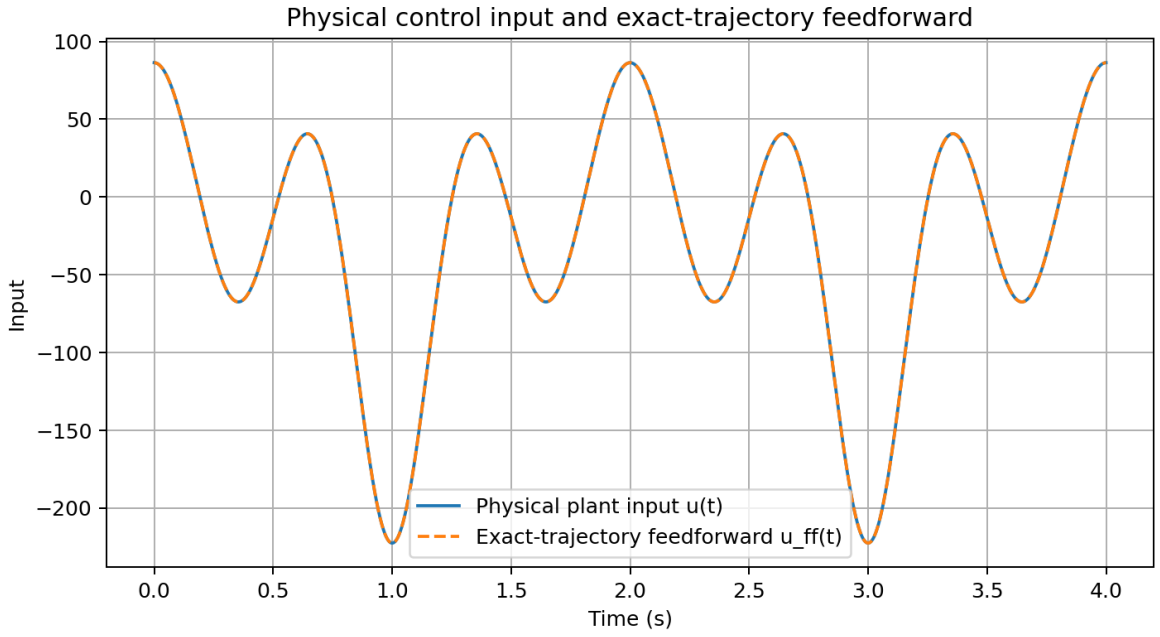


Figure 48. Physical plant input and exact feedforward input for the desired trajectory.

7.1 Integral-Identification Results

For the identification data, the true coefficients [p0,p1,p2,p3]=[0,0,-1,1] were estimated as [3.918961e-05, -3.876349e-05, -1.00058, 1.000858]. The relative l2 error is 7.332e-04. With this $\hat{f}$, S5 gives RMSE 1.143x10^{-4} and max|e| 2.653x10^{-4}, so the be-

havior approaches exact cancellation very closely without using a numerical second derivative of the measurement.

The coefficient error and the subsequent tracking error should be interpreted together. The integral regression recovers the polynomial coefficients closely enough that the residual cancellation error is small, and the resulting RMSE and maximum error approach the exact-model case. This links identification quality directly to closed-loop performance and supports the use of the integral formulation when numerical differentiation of measured signals would otherwise contaminate the estimate.

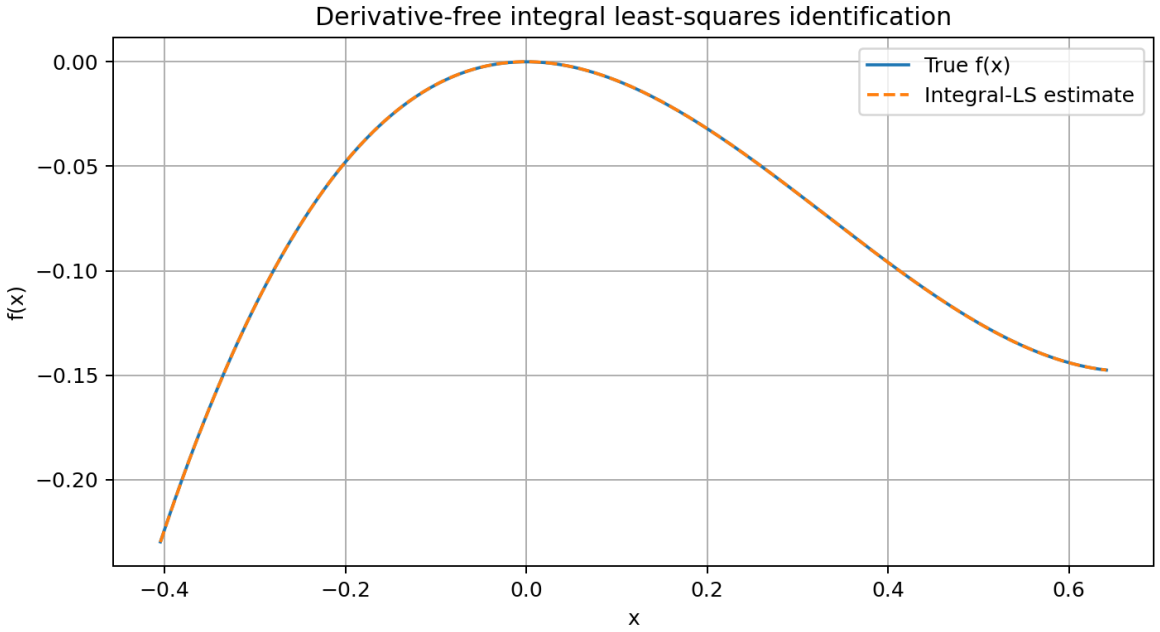


Figure 49. True and estimated nonlinearity using integral least squares.

7.2 Triangular Reference as a Stress Test

A triangular reference is used to test behavior under a nonsmooth command. No $\dddot{r}$ feedforward is used at slope discontinuities, and no "minimum energy" is computed. With the same gains, IAE=0.4638, RMSE=2.045, and peak |u| approximately 5.57x10^3 are obtained. The result shows that reference corners strongly excite derivative terms and that command prefiltering/reference shaping is required in practical applications.

The stress test exposes a different limitation from model mismatch. Even with the cancellation concept intact, a reference with slope discontinuities demands aggressive derivative-related action and produces a very large peak input. The poor RMSE and high peak |u| therefore arise from command regularity and controller realization, showing that exact model inversion does not remove the need for feasible reference design, command prefiltering, and actuator constraints.

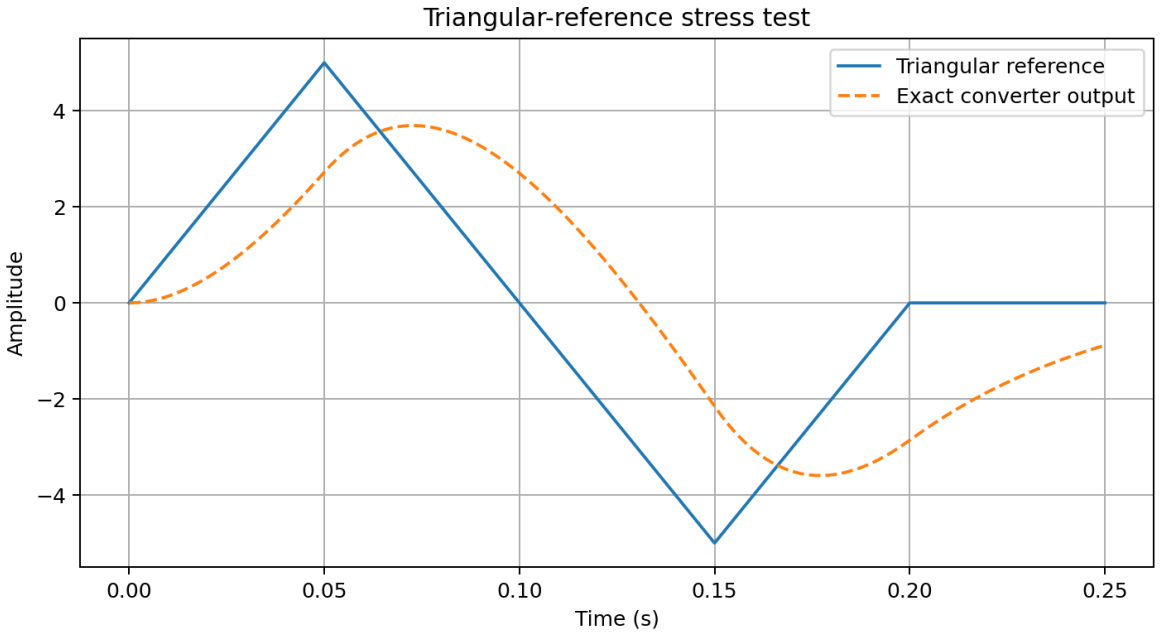


Figure 50. Stress test with a triangular reference.

8 Methodological and Numerical-Implementation Verification

The numerical implementation uses state integration rather than high-order finite differences. For L=2, the states x and $\dot{x}$ are integrated together with the controller integral state. This avoids repeated numerical differentiation of both x and u, which could amplify discretization errors in a finite-difference implementation.

The linear frequency response of the LES can be computed analytically from $1/s$^L and the selected C(s). For the nonlinear physical plant, a “transfer function” from a single finite experiment is generally not a signal-independent property.

In an application with actuator limits, saturation must be applied to the physical input u after the Converter. The identity $x^{(L)}$=v then ceases to hold during saturation, and anti-windup plus a separate stability analysis are required. For this reason, the baseline proof-of-equivalence scenario is run without saturation.

9 Limitations and Validity Domain

The method is theoretically strong but has clear limitations. It requires the nonlinearity to enter through the same channel as u, sufficient measurement/estimation of the states required by $\hat{f}$, a known nonzero input gain when generalized to b(x)u, and avoidance of regions in which the model or actuator cannot realize the required cancellation. By itself, it does not address nonminimum-phase internal dynamics, unmatched disturbances, delays, hysteresis, or nonsmooth nonlinearities.

Exact cancellation may also be sensitive to model mismatch. The proposed integral-LS estimate reduces model error in the polynomial case examined, but stability guarantees under noise/uncertainty require additional robust or adaptive design. The numerical results are therefore interpreted as verification of the principle rather than as a universal proof of robustness.

10 Conclusions

Identification of an unknown polynomial nonlinearity is formulated as integral linear regression, avoiding direct Lth-order numerical differentiation of measurements.

The numerical results confirm exact LES equivalence for known f, very small error when the integral-LS estimate is used, and the expected degradation under model mismatch or without cancellation. With this formulation, study is transformed from an unsupported output-transformation into a specific, clearly bounded, and mathematically correct application of exact feedback linearization with a practical identification procedure.

The numerical evidence therefore supports the method in a clear hierarchy: exact knowledge of f gives exact LES equivalence in the tested solver; a highly accurate identified $\hat{f}$ gives nearly the same tracking; coefficient mismatch produces a measurable residual error; and removal of the Converter gives substantially poorer tracking. The triangular-reference experiment adds that even exact cancellation cannot make an infeasible or nonsmooth command inexpensive. The method is consequently best understood as a precise structural linearization tool whose practical success still depends on model quality, reference smoothness, actuator capability, and robustness of the outer controller.

Relationship to the Complete Book

This paper is a self-contained study extracted and reformatted from the corresponding research chapter of *Nonlinear Control: From Model to Evidence*. The complete book provides the unified progression across all seven studies, cross-study synthesis, extended methodological context, and the MATLAB/Octave and Python software-companion architecture used for replication and further experimentation.

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